Local Divergences for Atanassov Intuitionistic Fuzzy Sets

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***Abstract*—The comparison of Atanassov intuitionistic fuzzy sets (AIF-sets) is a topic that has been widely studied due to its several applications in image segmentation or decision making, among other fields. Divergences for AIF-sets (AIF-divergences) were introduced as an adequate measure of comparison for AIF-sets. This study investigates a family of AIF-divergences that satisfies a local property. Such a property allows us to compute the divergence between AIF-sets pointwise. A characterization of those AIF-divergences satisfying the local property is provided. Several interesting properties of local divergence are also discussed. Some applications of these AIF-divergences in pattern recognition and decision making illustrate their utility.**

***Index Terms*—Atanassov intuitionistic fuzzy sets (AIF-sets), decision making, divergences, local property, pattern recognition.**

I. Introduction

Atanassov intuitionistic fuzzy set theory was introduced by Atanassov [1] as an extension of fuzzy set theory [33] and as an alternative to model situations in which fuzzy sets do not provide all the available information. While fuzzy sets just allow a membership degree, Atanassov intuitionistic fuzzy sets (AIF-sets) allow both a membership and nonmembership degrees [5]. AIF-sets have been proved to be a very powerful tool to model different real problems.

The comparison of objects described by fuzzy sets or any of their extension is a usual topic of research due to its applications in several areas, such as image segmentation [21], decision making [17], [32], or pattern recognition [16], [19].

Although the measures of comparison of fuzzy sets have been widely studied and several theoretical studies can be found in the literature [6], [12], the comparison of AIF-sets has not been studied so much. For this reason, in previous works [24], [26], we made an extensive study of different measures of comparison of AIF-sets, like distances and dissimilarities. We showed that dissimilarities could produce some counterintuitive measures that are not useful in applications, and for this reason, we introduced a class of divergences for AIF-sets (AIF-divergences), which is an extension of the divergences for the comparison of fuzzy sets [23]. AIF-divergences impose stronger conditions than dissimilarities, and for this reason, they avoid those counterintuitive situations.

In this study, we investigate a particular family of AIF-divergences that possess very interesting properties. This is the family of AIF-divergences that satisfies a local property that allows computation of AIF-divergence pointwise. We present a characterization and several properties of local AIF-divergences. In [26], we gave a method for building fuzzy divergences from AIF-divergences and, conversely, AIF-divergences from fuzzy divergences. Since local AIF-divergences can be seen as an extension of local fuzzy divergences [23], we investigate conditions under which such methods preserve locality. Local AIF-divergences are interesting not only from a theoretical, but from an applied point of view as well. We illustrate this fact showing how they could be applied in two diverse fields: pattern recognition and decision making. A preliminary version of some results of this study was presented in [25].

This paper is organized as follows. After this introduction, we provide an overview on the theory of AIF-sets, and we present AIF-divergences as measures of comparison of AIF-sets. Then, in Section III, we introduce the local property of AIF-divergences. For this, we first consider the local properties of fuzzy divergences, and then, we extend it to AIF-divergences. Then, we prove the characterization of local AIF-divergences, and we show that some of the usual AIF-divergences satisfy this local property. Section IV is devoted to investigating some of the properties of local AIF-divergences. Section V shows how to build local fuzzy divergences from local AIF-divergences and, conversely, local AIF-divergences from local fuzzy divergences. Then, some applications of local AIF-divergences in decision making and pattern recognition are discussed in Section VI, and we conclude the work with some comments and possible future lines of research in Section VII.

II. Atanassov Intuitionistic Fuzzy Sets

AIF-sets were introduced by Atanassov [1] as an extension of fuzzy sets [33]. A fuzzy set A is characterized by its membership function, such that $A(x)$ represents the degree to which x satisfies the property described by A. Atanassov realized that in some situations, fuzzy sets do not model adequately the available information. Therefore, he introduced an extension of fuzzy sets, called AIF-sets, that allowed to take into account not only the degree to which any element belongs to the set but also the degree to which it does not belong to the set. Formally, an AIF-set is defined by means of two functions $\mu_A, \nu_A : X \to [0,1]$, named membership and nonmembership functions, satisfying

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the constraint $\mu_A(x) + \nu_A(x) \leq 1$. This way, an AIF-set is defined by $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$.

If A is a fuzzy set, the nonmembership function coincides with one minus the membership function. However, for proper AIF-sets, the hesitation index, which is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, is nonzero, and it expresses the lack of knowledge on the membership of x to A. This index shows the usefulness of AIF-sets with respect to fuzzy sets: Fuzzy sets does not take into account the lack of information given by the hesitation index.

We denote by $FS(X)$ the set of all fuzzy sets on X and by $AIFS(X)$ the set of all AIF-sets on X. Trivially, $FS(X) \subset AIFS(X)$.

Usual operations on fuzzy sets are also defined for AIF-sets [1]–[4].

1) Union of A and B: $A \cup B = \{(x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)) : x \in X\}$, where $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\}$.
2) Intersection of A and B: $A \cap B = \{(x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)) : x \in X\}$, where $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$.
3) Complement of A: $A^c = \{(x, \nu_A(x), \mu_A(x)) : x \in X\}$.
4) A is a subset of B (denoted by $A \subseteq B$) if and only if for every $x \in X$, it holds that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

We have presented the original definitions given by Atanassov [1]–[4], since they are the most commonly used in the literature. However, there are generalizations based on t-norms and t-conorms (see, for example, [13]).

The comparison of fuzzy sets is an interesting topic of research [6], and several measures have been proposed in the past, like distances, dissimilarities [20], and divergences [23]. With the increasing popularity of the theory of AIF-sets, these measures were generalized to the comparison of AIF-sets. In a previous paper (see [26]), we have discussed several measures of comparison of AIF-sets that can be found in the literature. Dissimilarities are one of such measures; they are functions $D : AIFS(X) \times AIFS(X) \to \mathbb{R}$ satisfying the following axioms:

AIF-Diss.1: $D(A, A) = 0$ for every $A \in AIFS(X)$.

AIF-Diss.2: $D(A, B) = D(B, A)$ for every $A, B \in AIFS(X)$.

AIF-Diss.3: For every $A, B, C \in AIFS(X)$ such that $A \subseteq B \subseteq C$, it holds that $D(A, C) \geq \max(D(A, B), D(B, C))$.

Nevertheless, we have explained that such measures are not adequate in some situations because they could generate counterintuitive measures, as the next example shows.

Example 2.1 (see[26]): Consider a finite universe $X = \{x_1, \ldots, x_n\}$ and the function D_C defined, for every $A, B \in AIFS(X)$, by

$$D_{\mathrm{C}}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} |S_A(x_i) - S_{\mathrm{B}}(x_i)| \tag{1}$$

where $S_A(x_i) = |\mu_A(x_i) - \nu_A(x_i)|$ and $S_{\mathrm{B}}(x_i) = |\mu_B(x_i) - \nu_B(x_i)|$, for $i = 1, \ldots, n$. This measure was introduced by Chen [10], [11], and it is a dissimilarity measure. However, if $\mu_A(x_i) = \nu_A(x_i) = 0$ and $\mu_B(x_i) = \nu_B(x_i) = 0.5$ for every $i = 1, \ldots, n$, then $D_{\mathrm{C}}(A, B) = 0$. However, these two AIF-sets are distinctly different.

In order to avoid these counterintuitive examples, we introduced a new measure of comparison of AIF-sets that requires stronger conditions.

Divergences are measures of comparison for AIF-sets that model the following intuitive properties.

1) The divergence between two AIF-sets is nonnegative and symmetric.
2) The divergence between an Atanassov intuitionistic fuzzy set and itself must be zero.
3) The "more similar" two AIF-sets are, the lower is the divergence between them.

More precisely, we have the following.

Definition 2.2 (see[26, Def. 3.1]): AIF-divergence is a function $D_{\mathrm{AIF}} : AIFS(X) \times AIFS(X) \to \mathbb{R}$ satisfying the following axioms:

AIF-Diss.1: $D_{\mathrm{AIF}}(A, A) = 0$ for every $A \in AIFS(X)$.

AIF-Diss.2: $D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(B, A)$ for every $A, B \in AIFS(X)$.

AIF-Div.3: $D_{\mathrm{AIF}}(A \cap C, B \cap C) \leq D_{\mathrm{AIF}}(A, B)$ for every $A, B, C \in AIFS(X)$.

AIF-Div.4: $D_{\mathrm{AIF}}(A \cup C, B \cup C) \leq D_{\mathrm{AIF}}(A, B)$ for every $A, B, C \in AIFS(X)$.

Let us recall that the nonnegativity is not required in the definition. Nevertheless, it can be derived from the other axioms (see [26, Proposition 4.1]).

We have introduced AIF-divergences because dissimilarities could generate counterintuitive measures of comparison for AIF-sets. In fact, the next result establishes that every AIF-divergence is a dissimilarity, and therefore, AIF-divergences are more restrictive than dissimilarities since they require stronger conditions. However, sometimes, it can be an advantage, since the more restrictive the conditions are, the more "robust" the measure is, in the sense that it is less likely to result in counterintuitive examples. In fact, the dissimilarity of Chen defined in (1) is not an AIF-divergence.

Proposition 2.3 (see[26, Proposition 3.3]): Every AIF-divergence is a dissimilarity, and the converse does not hold in general.

Remark 2.4: If we restrict the AIF-divergence to the set of all fuzzy sets on X, that is, when we consider the restriction $D_{\mathrm{AIF}}|_{FS(X)} : FS(X) \times FS(X) \to \mathbb{R}$, the map $D_{\mathrm{AIF}}|_{FS(X)}$ is a usual divergence between fuzzy sets according to the definition proposed in [23].

III. Local Divergences for Atanassov Intuitionistic Fuzzy Sets

We have Already mentioned in the previous section that AIF-divergences are adequate measures of comparison for AIF-sets. In this section, we introduce a special family of AIF-divergences, which satisfy a local property, that will be very useful in several applications, as shall be shown in Section VI.

We split this section in three parts: First, we introduce the notion of locality for AIF-divergences and justify their use; second, we characterize them, and then, we conclude the section showing which of the usual measures of comparison of AIF-sets are local AIF-divergence.

A. Local Divergences for Atanassov Intuitionistic Fuzzy Sets: Definition and Justification

Consider a finite universe $X = \{x_1, \ldots, x_n\}$ and an AIF-divergence D_{AIF}. If A and B are two AIF-sets, and we consider $x_i \in X$, applying axiom AIF-Div.4, we know that $D_{\mathrm{AIF}}(A \cup \{x_i\}, B \cup \{x_i\}) \leq D_{\mathrm{AIF}}(A, B)$. In addition, it seems quite reasonable to assume that the difference between $D_{\mathrm{AIF}}(A \cup \{x_i\}, B \cup \{x_i\})$ and $D_{\mathrm{AIF}}(A, B)$ relies on the ith component. Thus, such a difference may only depend on $\mu_A(x_i), \nu_A(x_i)$ and $\mu_B(x_i), \nu_B(x_i)$. When this happens, the AIF-divergence is said to satisfy the local property.

Definition 3.1: Let D_{AIF} be an AIF-divergence. It is called local (or it is said to satisfy the local property) when for every $A, B \in AIFS(X)$ and every $x \in X$, it holds that

$$D_{\mathrm{AIF}}(A, B) - D_{\mathrm{AIF}}(A \cup \{x\}, B \cup \{x\}) = h_{\mathrm{AIF}}(\mu_A(x), \nu_A(x), \mu_B(x), \nu_B(x)). \quad (2)$$

This means that the change in the divergence only depends on what has been changed. As we will see in the next section, it is possible to provide a characterization of this kind of divergences in terms of the map $h_{\mathrm{AIF}} : [0, 1]^4 \to \mathbb{R}$.

Note that it is possible to find AIF-divergences that do not satisfy such property, as the one defined by Li *et al.* [18]:

$$D_O(A, B) = \frac{1}{\sqrt{2n}} \left(\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 \right)^{\frac{1}{2}}. \quad (3)$$

It is an AIF-divergence (see [26]), but it cannot be expressed as in (2).

The question now is: Why do we study this family of AIF-divergences? Are they really relevant only from the theoretical point of view or they are important for applications also? We now justify the main reasons for studying this family.

1) *Pointwise comparisons:* By definition, local AIF-divergences are those AIF-divergences that allow us to compare AIF-sets pointwise. This is quite relevant when dealing with applications like image processing. In that framework, sometimes, it is required to compare images pixel by pixel, or in other words, pointwise.
2) *Connection to restricted equivalence functions:* In [8] and [9], the notion of restricted equivalence functions was introduced as an extension of Fodor and Roubens fuzzy equivalences [14], and they were used to define similarity measures between fuzzy sets. We shall see in the next section that restricted equivalence functions can also be used to define local AIF-divergences.
3) *Computation complexity:* Obviously, when the cardinality of the referential set increases, the complexity of computation of any measure of comparison between AIF-sets (local and nonlocal AIF-divergence measures, distances, dissimilarities, etc.) also increases. However, the use of local AIF-divergences can make that computation easier. For example, after computing the difference between two AIF-sets A and B by means of a local AIF-divergence, if we need to compute the local AIF-divergence between A' and B, where the membership and nonmembership degrees of A' coincide with those of A in all but one of the elements, x_i, then using the local property, we can drastically simplify the computation. The reason is that, according to (2), we only need to compute the values: $h(\mu_{A'}(x_i), \nu_{A'}(x_i), \mu_B(x_i), \nu_B(x_i))$ and $h(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$. Obviously, this fact reduces considerably the complexity of the problem.

B. Characterization of Local Divergences for Atanassov Intuitionistic Fuzzy Sets

Now, we are going to provide a characterization of local AIF-divergence in terms of the properties satisfied by the function h_{AIF}.

Theorem 3.2: A map $D_{\mathrm{AIF}} : AIFS(X) \times AIFS(X) \to \mathbb{R}$ on a finite universe $X = \{x_1, \ldots, x_n\}$ is a local AIF-divergence if and only if there is a function $h_{\mathrm{AIF}} : \mathcal{D}^2 \to \mathbb{R}$, where $\mathcal{D}$ denotes the set $\mathcal{D} = \{(u, v) \in [0, 1]^2 : u + v \leq 1\}$, such that for every $A, B \in AIFS(X)$, it holds that

$$D_{\mathrm{AIF}}(A, B) = \sum_{i=1}^{n} h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) \quad (4)$$

and h_{AIF} satisfies the following properties:

AIF-loc.1: $h_{\mathrm{AIF}}(u, v, u, v) = 0$ for every $(u, v) \in \mathcal{D}$.

AIF-loc.2: $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = h_{\mathrm{AIF}}(v_1, v_2, u_1, u_2)$ for every $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$.

AIF-loc.3: If $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$, $w \in [0, 1]$ and $u_1 \leq w \leq v_1$, it holds that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\mathrm{AIF}}(u_1, u_2, w, v_2)$.

Moreover, if $\max\{u_2, v_2\} + w \leq 1$, it holds that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\mathrm{AIF}}(w, u_2, v_1, v_2)$.

AIF-loc.4: If $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$, $w \in [0, 1]$ and $u_2 \leq w \leq v_2$, it holds that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\mathrm{AIF}}(u_1, u_2, v_1, w)$.

Moreover, if $\max\{u_1, v_1\} + w \leq 1$, it holds that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\mathrm{AIF}}(u_1, w, v_1, v_2)$.

AIF-loc.5: If $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$ and $w \in [0, 1]$, then $h_{\mathrm{AIF}}(w, u_2, w, v_2) \leq h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2)$ if $\max\{u_2, v_2\} + w \leq 1$ and $h_{\mathrm{AIF}}(u_1, w, v_1, w) \leq h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2)$ if $\max\{u_1, v_1\} + w \leq 1$.

Proof: Let us assume, first of all, that D_{AIF} is a local AIF-divergence and prove that $D_{\mathrm{AIF}}(A, B)$ can be expressed as in (4), where h_{AIF} satisfies properties from AIF-loc.1 to AIF-loc.5. In order to prove that, we apply recursively (2): $D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(A \cup \{x_1\}, B \cup \{x_1\}) + h_{\mathrm{AIF}}(\mu_A(x_1), \nu_A(x_1), \mu_B(x_1),$

$\nu_B(x_1)) = D_{\text{AIF}}(A \cup \{x_1\} \cup \{x_2\}, B \cup \{x_1\} \cup \{x_2\}) + \sum_{i=1}^{2} h_{\text{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) = \cdots = D_{\text{AIF}}(X, X) + \sum_{i=1}^{n} h_{\text{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$.

Moreover, axiom AIF-Div.1 implies $D_{\text{AIF}}(X, X) = 0$, and consequently, $D_{\text{AIF}}(A, B)$ can be expressed as in (4).

Let us prove next that h_{AIF} fulfills properties from AIF-loc.1 to AIF-loc.5.

1) *AIF-loc.1:* Take $(u, v) \in \mathcal{D}$, and let us prove that $h_{\text{AIF}}(u, v, u, v) = 0$. For this, we define the AIF-set A by $\mu_A(x_i) = u$ and $\nu_A(x_i) = v$ for every $i = 1, \ldots, n$. Note that A is in fact an AIF-set since $\mu_A(x_i) + \nu_A(x_i) = u + v \leq 1$ for every $i = 1, \ldots, n$. Applying axiom AIF-Diss.1, $D_{\text{AIF}}(A, A) = 0$, and therefore, $0 = D_{\text{AIF}}(A, A) = \sum_{i=1}^{n} h_{\text{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_A(x_i), \nu_A(x_i)) = \sum_{i=1}^{n} h_{\text{AIF}}(u, v, u, v) = n h_{\text{AIF}}(u, v, u, v)$. Then, $h_{\text{AIF}}(u, v, u, v)$ must be 0.
2) *AIF-loc.2:* Let $(u_1, u_2), (v_1, v_2)$ be two elements in $\mathcal{D}$. Consider the AIF-sets A and B defined by $\mu_A(x_i) = u_1$, $\nu_A(x_i) = u_2$, $\mu_B(x_i) = v_1$, and $\nu_B(x_i) = v_2$ for any $i = 1, \ldots, n$. Using axiom AIF-Diss.2 and (4), we obtain the following: $n h_{\text{AIF}}(u_1, u_2, v_1, v_2) = \sum_{i=1}^{n} h_{\text{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) = D_{\text{AIF}}(A, B) = D_{\text{AIF}}(B, A) = \sum_{i=1}^{n} h_{\text{AIF}}(\mu_B(x_i), \nu_B(x_i), \mu_A(x_i), \nu_A(x_i)) = n h_{\text{AIF}}(v_1, v_2, u_1, u_2)$. Thus, $h_{\text{AIF}}(u_1, u_2, v_1, v_2) = h_{\text{AIF}}(v_1, v_2, u_1, u_2)$.
3) *AIF-loc.3:* Consider $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$ and $w \in [0, 1]$ such that $u_1 \leq w \leq v_1$, and let us define the AIF-sets A and B by $\mu_A(x_i) = u_1$, $\nu_A(x_i) = u_2$, $\mu_B(x_i) = v_1$, and $\nu_B(x_i) = v_2$, for every $i = 1, \ldots, n$. Consider two cases.
   a) On one hand, we are going to prove that $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(u_1, u_2, w, v_2)$. To see this, consider the AIF-set C defined by $\mu_C(x_i) = w$ and $\nu_C(x_i) = 0$ for $i = 1, \ldots, n$. Then, the AIF-sets $A \cap C$ and $B \cap C$ are given by $A \cap C = A$ and $B \cap C = \{(x_i, \mu_C(x_i), \nu_B(x_i)) : i = 1, \ldots, n\}$.
   Using axiom AIF-Div.3, we see that $D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(A \cap C, B \cap C) = D_{\text{AIF}}(A, B \cap C)$, and then, (4) implies that $n h_{\text{AIF}}(u_1, u_2, v_1, v_2) = D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(A \cap C, B \cap C) = n h_{\text{AIF}}(u_1, u_2, w, v_2)$. Hence, $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(u_1, u_2, w, v_2)$.
   b) Now, we prove that $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(w, u_2, v_1, v_2)$ holds when $\max(u_2 + w, v_2 + w) \leq 1$. Consider the AIF-set C defined by $\mu_C(x_i) = w$ and $\nu_C(x_i) = \max(u_2, v_2)$, for $i = 1, \ldots, n$. Note that C is an AIF-set because $\mu_C(x_i) + \nu_C(x_i) = \max(u_2 + w, v_2 + w) \leq 1$, for $i = 1, \ldots, n$. Using axiom AIF-Div.4, we deduce that $D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(A \cup C, B \cup C)$. Moreover, the AIF-sets $A \cup C$ and $B \cup C$ are given by $A \cup C = \{(x_i, \mu_C(x_i), \nu_A(x_i) : i = 1, \ldots, n\}$ and $B \cup C = B$.
   Then, $D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(A \cup C, B)$. This, together with (4), implies that $n h_{\text{AIF}}(u_1, u_2, v_1, v_2) = D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(A \cup C, B \cup C) = n h_{\text{AIF}}(w, u_2, v_1, v_2)$. Hence, $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(w, u_2, v_1, v_2)$.
4) *AIF-loc.4:* The proof is analogous to that of AIF-loc.3.
5) *AIF-loc.5:* Consider $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$ and $w \in [0, 1]$.
   Let us assume that $\max(u_2, v_2) + w \leq 1$, and we consider the AIF-sets A, B, C, and D given by $A = \{(x, u_1, u_2) : x \in X\}$, $B = \{(x, v_1, v_2) : x \in X\}$, $C = \{(x, w, u_2) : x \in X\}$ and $D = \{(x, w, v_2) : x \in X\}$. Using [26, Proposition 4.5], it follows that $D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(C, D)$, and applying (4), we deduce that $n h_{\text{AIF}}(u_1, u_2, v_1, v_2) = D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(C, D) = n h_{\text{AIF}}(w, u_2, w, v_2)$. Thus, $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(w, u_2, w, v_2)$.
   If we assume now that $\max(u_1, v_1) + w \leq 1$, and we consider the AIF-sets: $A = \{(x, u_1, u_2) : x \in X\}$, $B = \{(x, v_1, v_2) : x \in X\}$, $C = \{(x, u_1, w) : x \in X\}$ and $D = \{(x, v_1, w) : x \in X\}$. Applying [26, Proposition 4.5], we know that $D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(C, D)$. Using (4), we obtain that $n h_{\text{AIF}}(u_1, u_2, v_1, v_2) = D_{\text{AIF}}(A, B) \geq D_{\text{AIF}}(C, D) = n h_{\text{AIF}}(u_1, w, v_1, w)$. Thus, $h_{\text{AIF}}(u_1, u_2, v_1, v_2) \geq h_{\text{AIF}}(u_1, w, v_1, w)$.

Summarizing, we have proven that if D_{AIF} is an AIF-divergence satisfying the local property, then $D_{\text{AIF}}(A, B)$ can be expressed as in (4), where the function h_{AIF} satisfies AIF-loc.1–AIF-loc.5.

Let us prove the converse. That is, we are going to prove that if a function D_{AIF} is defined by (4), where h_{AIF} fulfills properties AIF-loc.1–AIF-loc.5, then D_{AIF} is a local AIF-divergence.

First of all, let us prove that D_{AIF} is an AIF-divergence, i.e., that it satisfies axioms AIF-Diss.1, AIF-Diss.2, AIF-Div.3, and AIF-Div.4.

1) *AIF-Diss.1:* Let A be an AIF-set. Then, $D_{\text{AIF}}(A, A) = 0$ because $D_{\text{AIF}}(A, A)$ is equal to $\sum_{i=1}^{n} h_{\text{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_A(x_i), \nu_A(x_i)) = 0$, since AIF-loc.1 implies that $h_{\text{AIF}}(u, v, u, v) = 0$ for every $(u, v) \in \mathcal{D}$, and in particular $(\mu_A(x_i), \nu_A(x_i)) \in \mathcal{D}$ for $i = 1, \ldots, n$.
2) *AIF-Diss.2:* Let A and B be two AIF-sets, and let us prove that $D_{\text{AIF}}(A, B) = D_{\text{AIF}}(B, A)$. Applying AIF-loc.2, for every $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$, it holds that
$$h_{\text{AIF}}(u_1, u_2, v_1, v_2) = h_{\text{AIF}}(v_1, v_2, u_1, u_2).$$
   Hence, $D_{\text{AIF}}(A, B) = D_{\text{AIF}}(B, A)$.
3) *AIF-Div.3 and AIF-Div.4:* Let A, B, and C be three AIF-sets, and let us show that $D_{\text{AIF}}(A, B) \geq \max(D_{\text{AIF}}(A \cup C, B \cup C), D_{\text{AIF}}(A \cap C, B \cap C))$. Consider the following subsets of X: $P_1 = \{x \in X : \mu_A(x), \mu_B(x) \leq \mu_C(x)\}$, $P_2 = \{x \in X : \mu_A(x) \leq \mu_C(x) < \mu_B(x)\}$, $P_3 = \{x \in X : \mu_B(x) \leq \mu_C(x) < \mu_A(x)\}$, $P_4 = \{x \in X : \mu_C(x) < \mu_A(x), \mu_B(x)\}$, $Q_1 = \{x \in X : \nu_A(x), \nu_B(x) \leq \nu_C(x)\}$, $Q_2 = \{x \in X : \nu_A(x) \leq \nu_C(x) < \nu_B(x)\}$, $Q_3 = \{x \in X : \nu_B(x) \leq \nu_C(x) < \nu_A(x)\}$, and $Q_4 = \{x \in X : \nu_C(x) < \nu_A(x), \nu_B(x)\}$.
   Thus, $X = \bigcup_{i=1}^{4} \bigcup_{j=1}^{4} (P_i \cap Q_j)$. Using properties from AIF-loc.1 to AIF-loc.5, it can be easily proven that, for every $i, j \in \{1, \ldots, 4\}$ and $x \in P_i \cap Q_j$, it holds that $h_{\text{AIF}}(\mu_{A\cup C}(x), \nu_{A\cup C}(x), \mu_{B\cup C}(x), \nu_{B\cup C})$ and $h_{\text{AIF}}(\mu_{A\cap C}(x), \nu_{A\cap C}(x), \mu_{B\cap C}(x), \nu_{B\cap C})$ are lower than $h_{\text{AIF}}(\mu_A(x), \nu_A(x), \mu_B(x), \nu_B(x))$. Thus, D_{AIF} satisfies

both AIF-Div.3 and AIF-Div.4, and therefore, it is an AIF-divergence.

It only remains to show that $D_{\rm AIF}$ is local, but this holds cx trivially, taking into account that, for $j = 1, \ldots, n$, $D_{\rm AIF}(A, B) - D_{\rm AIF}(A \cup \{x_j\}, B \cup \{x_j\}) = \sum_{i=1}^{n} h_{\rm AIF}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) - \sum_{i \neq j} h_{\rm AIF}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) - h_{\rm AIF}(1, 1, 0, 0) = h_{\rm AIF}(\mu_A(x_j), \nu_A(x_j), \mu_B(x_j), \nu_B(x_j))$. We conclude that $D_{\rm AIF}$ is a local AIF-divergence. ■

This theorem allows us to characterize local AIF-divergences by means of a function $h_{\rm AIF}$ that satisfies conditions from AIF-loc.1 to AIF-loc.5. Thus, given a function $D : AIFS(X) \times AIFS(X) \to \mathbb{R}$, in order to check whether it is a local AIF-divergence or not, it is enough to prove that it can be expressed as in (4), and its associated function $h_{\rm AIF}$ satisfies such conditions.

Remark 3.3: In the particular case, $h_{\rm AIF}$ only depends on the values of the first and third components; we have a local divergence between fuzzy sets, since it is characterized (see [23, Proposition 3.4]) by a function $h : [0, 1] \times [0, 1] \to \mathbb{R}$ such that $D(A, B) = \sum_{i=1}^{n} h(A(x), B(x))$, and it satisfies the following properties:

loc.1: $h(u, u) = 0$ for every $u \in [0, 1]$.

loc.2: $h(u, v) = h(v, u)$ for every $(u, v) \in [0, 1]^2$.

loc.3: $h(u, w) \geq \max(h(u, v), h(v, w))$ for every $u, v, w \in [0, 1]$ such that $u < v < w$.

C. Examples of Local Divergences for Atanassov Intuitionistic Fuzzy Sets

Once we have characterized local AIF-divergences in terms of the function $h_{\rm AIF}$, we are going to see several examples. We have already seen one example of an AIF-divergence that is not local: Li's AIF-divergence defined in (3).

The two main AIF-divergences we can find in the literature are Hamming and Hausdorff distances, defined by $l_{\rm IFS}(A, B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|$ and $d_{\rm H}(A, B) = \sum_{i=1}^{n} \max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|)$.

We have already proved (see [26]) that both measures are AIF-divergences. In fact, if we consider the functions $h_{l_{\rm IFS}}(u_1, u_2, v_1, v_2) = |u_1 - v_1| + |u_2 - v_2| + |u_1 + u_2 - v_1 - v_2|$, and $h_{d_{\rm H}}(u_1, u_2, v_1, v_2) = \max(|u_1 - v_1|, |u_2 - v_2|)$, both Hamming and Hausdorff distances can be expressed, respectively, by

$$l_{\rm IFS}(A, B) = \sum_{i=1}^{n} h_{l_{\rm IFS}}(\mu_A(x_i), \mu_B(x_i), \nu_A(x_i), \nu_B(x_i)) \tag{5}$$

$$d_{\rm H}(A, B) = \sum_{i=1}^{n} h_{d_{\rm H}}(\mu_A(x_i), \mu_B(x_i), \nu_A(x_i), \nu_B(x_i)). \tag{6}$$

Thus, both Hamming and Hausdorff distances are local AIF-divergences.

A recent paper of Szmidt and Kacprzyk [30] presents a survey of several measures of comparison of AIF-sets. In [26], we have investigated the cases that are AIF-divergences. Now, we are going to see which of them satisfy the local property.

We start considering the AIF-divergences defined by Hong and Kim [15]:

$$D_{HK}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \tag{7}$$

$$D_L(A, B) = \frac{1}{4n} \sum_{i=1}^{n} |S_A(x_i) - S_B(x_i)| + |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \tag{8}$$

where $S_A(x_i) = |\mu_A(x_i) - \nu_A(x_i)|$ and $S_B(x_i) = |\mu_B(x_i) - \nu_B(x_i)|$. Both AIF-divergences are local, and their associated functions h_{HK} and h_L are, respectively, given by $h_{HK}(u_1, u_2, v_1, v_2) = \frac{1}{2n}(|u_1 - v_1| + |u_2 - v_2|)$ and $h_L(u_1, u_2, v_1, v_2) = \frac{1}{4n}(|u_1 - u_2 - v_1 + v_2| + |u_1 - v_1| + |u_2 - v_2|)$.

The AIF-divergences of Mitchell [22] (D_{HB}^p) and Liang and Shi (D_e^p and D_h^p) are defined by $D_{HB}^p(A, B) = \frac{1}{\sqrt[p]{n}}((\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |\nu_A(x_i) - \nu_B(x_i)|^p)^{\frac{1}{p}})$, $D_e^p(A, B) = \frac{1}{2\sqrt[p]{n}}(\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)^p)^{\frac{1}{p}}$, and $D_h^p(A, B) = \frac{1}{\sqrt[p]{3n}}\Big(\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\mu_A(x_i) - \mu_B(x_i) - \nu_A(x_i) + \nu_B(x_i)| + |\mu_A(x_i) - \mu_B(x_i) + \nu_A(x_i) - \nu_B(x_i)|\Big)^{\frac{1}{p}}$.

These AIF-divergences are not local except for the trivial case with $p = 1$. In fact, for $p = 1$, D_{HB}^1 coincides with the AIF-divergence D_{HK} given by Hong and Kim, which we have already seen that satisfies the local property.

IV. Properties of the Local Divergences for Atanassov Intuitionistic Fuzzy Sets

We have introduced local AIF-divergences as particular cases of AIF-divergences that can be computed pointwise. We devote this section to show several interesting properties satisfied by local AIF-divergences.

First of all, let us focus on a property that was introduced in [26] that some AIF-divergences may satisfy:

AIF-Div.5: $D_{\rm AIF}(A, B) = D_{\rm AIF}(A^c, B^c)$ for every $A, B \in AIFS(X)$.

This property becomes important since, if it is satisfied, the axioms AIF-Div.3 and AIF-Div.4 are equivalent.

Proposition 4.1 (see[26, Proposition 4.4]): Let $D_{\rm AIF} : AIFS(X) \times AIFS(X) \to \mathbb{R}$ be a function satisfying AIF-Diss.1, AIF-Diss.2, and AIF-Div.5. Then, it satisfies AIF-Div.3 if and only if it satisfies AIF-Div.4.

Consequently, if a function $D_{\rm AIF}$ satisfies AIF-Diss.1, AIF-Dist.2, and AIF-Div.5, in order to prove that it is an AIF-divergence, it is enough to check whether it satisfies either AIF-Div.3 or AIF-Div.4.

In case of local AIF-divergences, the condition AIF-Div.5 can be written in terms of the function $h_{\rm AIF}$.

Proposition 4.2: Let $D_{\rm AIF}$ be a local AIF-divergence. Then, $D_{\rm AIF}$ satisfies property AIF-Div.5 if and only if

$$h_{\rm AIF}(u_1, u_2, v_1, v_2) = h_{\rm AIF}(u_2, u_1, v_2, v_1) \tag{9}$$

for every $(u_1, u_2), (v_1, v_2) \in \mathcal{D} = \{(u, v) \in [0,1]^2 : u + v \leq 1\}$.

Proof: Assume that D_{AIF} satisfies axiom AIF-Div.5, i.e., $D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(B, A)$ for every $A, B \in AIFS(X)$. Consider $(u_1, u_2), (v_1, v_2) \in \mathcal{D}$ and define the AIF-sets A and B by $A = \{(x, u_1, u_2) : x \in X\}$ and $B = \{(x, v_1, v_2) : x \in X\}$.

Applying AIF-Div.5, it holds that $D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(A^c, B^c)$. Using (4), we find that $nh_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(A^c, B^c) = nh_{\mathrm{AIF}}(u_2, u_1, v_2, v_1)$.

Thus, $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = h_{\mathrm{AIF}}(u_2, u_1, v_2, v_1)$.

Conversely, assume that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = h_{\mathrm{AIF}}(u_2, u_1, v_2, v_1)$ for every elements (u_1, u_2) and (v_1, v_2) in $\mathcal{D}$. Let A and B be two AIF-sets. Then, for every $i = 1, \ldots, n$, it holds that $h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) = h_{\mathrm{AIF}}(\nu_A(x_i), \mu_A(x_i), \nu_B(x_i), \mu_B(x_i))$, and therefore, $D_{\mathrm{AIF}}(A, B) = D_{\mathrm{AIF}}(A^c, B^c)$. ■

According to Propositions 4.1 and 4.2, in order to prove that a function h_{AIF} satisfying (9) defines a local AIF-divergence, it is enough to prove that it satisfies AIF-loc.1, AIF-loc.2, AIF-loc.5 and either AIF-loc.3 or AIF-loc.4, because both are equivalent.

In Section III-C, we have shown that both Hamming and Hausdorff distances are local. In fact, both satisfy the axiom AIF-Div.5 (see [26]), but we could also prove it by means of the previous result, since functions $h_{\mathrm{l_{IFS}}}$ and $h_{\mathrm{d_H}}$ defined in (5) and (6) satisfy (9).

Our next result is an extension of [23, Propositions 3.10 and 3.11] to AIF-sets, and it ensures that the divergence takes the maximum value when the AIF-sets are crisp.

Proposition 4.3: Consider a local AIF-divergence D_{AIF} and any two crisp sets V and Z. Then, $D_{\mathrm{AIF}}(V, V^c) = D_{\mathrm{AIF}}(Z, Z^c)$. In addition, if $A, B \in AIFS(X)$, then $D_{\mathrm{AIF}}(A, B) \leq D_{\mathrm{AIF}}(Z, Z^c)$.

Proof: Note that $h_{\mathrm{AIF}}(1, 0, 0, 1) = h_{\mathrm{AIF}}(0, 1, 1, 0)$, and therefore, $D_{\mathrm{AIF}}(V, V^c) = nh_{\mathrm{AIF}}(1, 0, 0, 1) = D_{\mathrm{AIF}}(Z, Z^c)$.

Now, taking into account that $h_{\mathrm{AIF}}(1, 0, 0, 1) \geq h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2)$, since $h_{\mathrm{AIF}}(1, 0, 0, 1) \geq h_{\mathrm{AIF}}(u_1, 0, 0, 1) \geq h_{\mathrm{AIF}}(u_1, u_2, 0, 1) \geq h_{\mathrm{AIF}}(u_1, u_2, 0, v_2) \geq h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2)$, it holds that $D_{\mathrm{AIF}}(A, B) = \sum_{i=1}^{n} h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) \leq \sum_{i=1}^{n} h_{\mathrm{AIF}}(1, 0, 0, 1) = D_{\mathrm{AIF}}(Z, Z^c)$. ■

In [26, Proposition 4.14], we introduced a method that is useful to generate AIF-divergences from other AIF-divergences. It is based on a nondecreasing function ϕ satisfying $\phi(0) = 0$. In such a case, if D_{AIF} is an AIF-divergence, the function D^{ϕ}_{AIF} defined by $D^{\phi}_{\mathrm{AIF}}(A, B) = \phi(D_{\mathrm{AIF}}(A, B))$ is also an AIF-divergence. In this case, although D_{AIF} is a local AIF-divergence, D^{ϕ}_{AIF} may not be local. For instance, if ϕ is not a linear function, D^{ϕ}_{AIF} is not local. Nevertheless, it is possible to prove a similar result.

Proposition 4.4: Let D_{AIF} be a local AIF-divergence, and let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function satisfying $\phi(0) = 0$. In such a case, the function D^{ϕ}_{AIF}, defined by $D^{\phi}_{\mathrm{AIF}}(A, B) = \sum_{i=1}^{n} \phi(h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)))$, is a local AIF-divergence.

Proof: On one hand, since $\phi(0) = 0$, it holds that $\phi(h_{\mathrm{AIF}}(u, v, u, v)) = \phi(0) = 0$ for any $(u, v) \in \mathcal{D}$. Then, $\phi \circ h_{\mathrm{AIF}}$ satisfies AIF-loc.1. Furthermore, since h_{AIF} satisfies AIF-loc.2: $\phi \circ h_{\mathrm{AIF}}(u_1, v_1, u_2, v_2) = \phi(h_{\mathrm{AIF}}(u_1, v_1, u_2, v_2)) = \phi(h_{\mathrm{AIF}}(v_1, u_1, v_2, u_2)) = \phi \circ h_{\mathrm{AIF}}(v_1, u_1, v_2, u_2)$.

Then, $\phi \circ h_{\mathrm{AIF}}$ also satisfies AIF-loc.2. To prove that it also fulfills AIF-loc.3–AIF-loc.5, it is enough to note that h_{AIF} does satisfy them and that ϕ is a nondecreasing function. Then, using Theorem 3.2, D^{ϕ}_{AIF} a local AIF-divergence. ■

This result shows that if $h_{\mathrm{AIF}} : \mathcal{D}^2 \to \mathbb{R}$ is a function satisfying properties AIF-loc.1–AIF-loc.5, and φ is a nondecreasing function from $[0, \infty)$ to $[0, \infty)$ such that $\varphi(0) = 0$, the composition $\varphi \circ h_{\mathrm{AIF}}$ also satisfies properties AIF-loc.1–AIF-loc.5, and therefore, it defines a local AIF-divergence.

The next result relates local AIF-divergences and real distances.

Proposition 4.5: Consider a distance $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$u < v < w \Rightarrow \max(d(u, v), d(v, w)) \leq d(u, w). \qquad (10)$$

Then, for every nondecreasing function $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\phi(0, 0) = 0$, the function $D_{\mathrm{AIF}} : AIFS(X) \times AIFS(X) \to \mathbb{R}$ defined by the following equation is a local AIF-divergence:

$$D_{\mathrm{AIF}}(A, B) = \sum_{i=1}^{n} \phi(d(\mu_A(x_i), \mu_B(x_i)), d(\nu_A(x_i), \nu_B(x_i))). \qquad (11)$$

Proof: Using Theorem 3.2, it is enough to prove that the function $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = \phi(d(u_1, v_1), d(u_2, v_2))$ satisfies properties from AIF-loc.1 to AIF-loc.5.

AIF-loc.1: Since d is a distance, $d(x, x) = 0$. Then, for any $(x, y) \in \mathcal{D}$ $h_{\mathrm{AIF}}(x, y, x, y) = \phi(d(x, x), d(y, y)) = \phi(0, 0) = 0$.

AIF-loc.2: Since any distance is symmetric, for any $(x_1, x_2), (y_1, y_2) \in \mathcal{D}$, it holds that $h_{\mathrm{AIF}}(x_1, x_2, y_1, y_2) = \phi(d(x_1, y_1), d(x_2, y_2)) = \phi(d(y_1, x_1), d(y_2, x_2)) = h_{\mathrm{AIF}}(y_1, y_2, x_1, x_2)$.

AIF-loc.3: Consider $(x_1, x_2), (y_1, y_2) \in \mathcal{D}$ and z such that $x_1 \leq z \leq y_1$. Applying (10) and the monotonicity of ϕ, we obtain that $h_{\mathrm{AIF}}(x_1, x_2, y_1, y_2) = \phi(d(x_1, y_1), d(x_2, y_2)) \geq \phi(d(x_1, z), d(x_2, y_2)) = h_{\mathrm{AIF}}(x_1, x_2, z, y_2)$.

Moreover, if $\max(x_2, y_2) + z \leq 1$, it holds that $h_{\mathrm{AIF}}(x_1, x_2, y_1, y_2) = \phi(d(x_1, y_1), d(x_2, y_2)) \geq \phi(d(z, y_1), d(x_2, y_2)) = h_{\mathrm{AIF}}(z, x_2, y_1, y_2)$.

AIF-loc.4: The proof is analogous to the one of AIF-loc.3.

AIF-loc.5: Consider $(x_1, x_2), (y_1, y_2) \in \mathcal{D}$ and $z \in [0, 1]$ such that $\max\{x_2, y_2\} + z \leq 1$. Since $d(z, z) = 0$, it holds that $h_{\mathrm{AIF}}(z, x_2, z, y_2) = \phi(d(z, z), d(x_2, y_2) \leq \phi(d(x_1, y_1), d(x_2, y_2)) = h_{\mathrm{AIF}}(x_1, x_2, y_1, y_2)$.

Thus, h_{AIF} satisfies AIF-loc.1–AIF-loc.5, and therefore, Theorem 3.2 assures that D_{AIF} is a local AIF-divergence. ■

This proposition shows how to use distances to define local AIF-divergences. Let us see an example of application of this result.

Example 4.6: Consider the distance d defined by $d(u, v) = |u - v|$, and the nondecreasing function $\phi(u, v) = \frac{u+v}{2n}$ that trivially satisfies $\phi(0, 0) = 0$. Then, applying the previous result, the function D_{AIF} defined by (11) is a local AIF-divergence. In fact, if we input the values of ϕ and d, D_{AIF} becomes $D_{\mathrm{AIF}}(A, B) = \frac{1}{2n}\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|$.

We have obtained the AIF-divergence defined by Hong and Kim [see (7)].

To conclude this section, we are going to explore the connection between local AIF-divergences and restricted equivalence functions [8], [9]. A function $REF : [0,1]^2 \to [0,1]$ is a restricted equivalence function if it satisfies the following properties.

1) $REF(x,y) = REF(y,x)$ for any $x, y \in [0,1]$.
2) $REF(x,y) = 1$ if and only if $x = y$.
3) $REF(x,y) = 0$ if and only if $x = 1, y = 0$ or $x = 0, y = 1$.
4) $REF(x,y) = REF(c(x), c(y))$ for any $x, y \in [0,1]$ and strong negation c.
5) For any $x, y, z \in [0,1]$ such that $x \le y \le z$, $REF(x, y) \ge REF(x,z)$ and $REF(y,z) \ge REF(x,z)$.

It is known that any restricted equivalence function is in particular a fuzzy equivalence in the sense of Fodor and Roubens [14] (see [8, Th. 6]). In [8] and [9], it is proven that restricted equivalence functions can be used to define fuzzy similarity measures by means of an aggregation function. Next, we see how restricted equivalence function can be used to define local AIF-divergences.

Proposition 4.7: Let REF be a restricted equivalence function, and let $f : [0,1]^2 \to [0,\infty)$ be a componentwise increasing function with $f(0,0) = 0$. Then, the function $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = f(1 - REF(u_1,v_1), 1 - REF(u_2,v_2))$ satisfies properties AIF-loc.1–AIF-loc.5, and therefore, it defines a local AIF-divergence.

Proof: Let us prove that h_{AIF} satisfies properties AIF-loc.1–AIF-loc.5.

AIF-loc.1: $h_{\mathrm{AIF}}(u,v,u,v) = f(1 - REF(u,u), 1 - REF(v,v)) = f(0,0) = 0$.

AIF-loc.2: Taking into account that REF is commutative, $h_{\mathrm{AIF}}(u_1,v_1,u_2,v_2) = f(1 - REF(u_1,u_2), 1 - REF(v_1,v_2)) = f(1 - REF(u_2,u_1), 1 - REF(v_2,v_1)) = h_{\mathrm{AIF}}(u_2, v_2, u_1, v_1)$.

AIF-loc.3: Let $\omega \in [u_1, v_1]$. Then, it holds that $h_{\mathrm{AIF}}(u_1, u_2, v_1, v_2) = f(1 - REF(u_1,v_1), 1 - REF(u_2,v_2)) \ge f(1 - REF(u_1,\omega), 1 - REF(u_2,v_2)) = h_{\mathrm{AIF}}(u_1,u_2,\omega,v_2)$.

Furthermore, if $\max\{u_1,v_1\} + \omega \le 1$, it holds that $h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2) = f(1 - REF(u_1,v_1), 1 - REF(u_2,v_2)) \ge f(1 - REF(u_1,v_1), 1 - REF(\omega,v_2)) = h_{\mathrm{AIF}}(u_1,\omega,v_1,v_2)$.

AIF-loc.4: The proof is analogous to that of AIF-loc.3.

AIF-loc.5: Taking into account that $REF(\omega,\omega) = 0$ and f is componentwise increasing, it holds that $h_{\mathrm{AIF}}(\omega,u_2,\omega,v_2) = f(1 - REF(\omega,\omega), 1 - REF(u_2,v_2)) \le f(1 - REF(u_1,v_1), 1 - REF(u_2,v_2)) = h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2)$. ■

V. Local Divergences for Atanassov Intuitionistic Fuzzy Sets Versus Local Divergences

In [26], we have introduced a method that allows us to build divergences from AIF-divergences and, conversely, AIF-divergences from divergences. In this section, our aim is to investigate whether these methods preserve the local property or not. For this, first of all, we describe those methods, and then, we study the role of locality.

We start explaining how a divergence can be defined from an AIF-divergence. This is quite easy because if we restrict an AIF-divergence to $FS(X) \times FS(X)$, it becomes a fuzzy divergence.

Proposition 5.1 (see[26, Proposition 4.3]): Consider an AIF-divergence D_{AIF}. Then, the function $D : FS(X) \times FS(X) \to \mathbb{R}$ defined by

$$D(A,B) = D_{\mathrm{AIF}}(A,B) \text{ for every } A, B \in FS(X) \qquad (12)$$

is a divergence for fuzzy sets.

Conversely, from a divergence, it is possible to define an AIF-divergence.

Proposition 5.2 (see[26, Proposition 4.7]): Consider two fuzzy divergences D_1 and D_2 and a function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ that is nondecreasing on each component and satisfies $f(0,0) = 0$. Then, the function $D_{\mathrm{AIF}} : AIFS(X) \times AIFS(X) \to \mathbb{R}$ defined, for every $A, B \in AIFS(X)$, by

$$D_{\mathrm{AIF}}(A,B) = f(D_1(\mu_A,\mu_B), D_2(\nu_A,\nu_B)) \qquad (13)$$

is an AIF-divergence. In particular, it is possible to consider $D_1 = D_2$, and then, (13) becomes $D_{\mathrm{AIF}}(A,B) = f(D(\mu_A,\mu_B), D(\nu_A,\nu_B))$.

From now on, we consider divergences and AIF-divergences satisfying the local property, and we investigate if this property is preserved by the methods presented in Propositions 5.1 and 5.2. Next proposition shows that a local fuzzy divergence can be defined from a local AIF-divergence using the same method as in Proposition 5.1.

Proposition 5.3: Consider a local AIF-divergence D_{AIF}. The fuzzy divergence introduced in Proposition 5.1 is a local fuzzy divergence.

Proof: If D_{AIF} is a local AIF-divergence, it can be expressed by $D_{\mathrm{AIF}}(A,B) = \sum_{i=1}^{n} h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i))$.

Now, if A and B are fuzzy sets and we compute the fuzzy divergence between them, we obtain that $D(A,B) = D_{\mathrm{AIF}}(A,B) = \sum_{i=1}^{n} h_{\mathrm{AIF}}(A(x_i), 1 - A(x_i), B(x_i), 1 - B(x_i)) = \sum_{i=1}^{n} h(A(x_i), B(x_i))$, where $h(u,v) = h_{\mathrm{AIF}}(u, 1-u, v, 1-v)$. Let us verify that h satisfies loc.1–loc.3. First of all, applying AIF-loc.1 to h_{AIF}, it holds that $h(u,u) = h_{\mathrm{AIF}}(u, 1-u, u, 1-u) = 0$. Now, using AIF-loc.2, $h(u,v) = h_{\mathrm{AIF}}(u, 1-u, v, 1-v) = h_{\mathrm{AIF}}(v, 1-v, u, 1-u) = h(v,u)$.

Finally, it only remains to check loc.3. Consider $u < v < w$. Applying first AIF-loc.3 and later AIF-loc.4, it holds that $h(u,w) = h_{\mathrm{AIF}}(u, 1-u, w, 1-w) \ge \max(h_{\mathrm{AIF}}(u, 1-u, v, 1-w), h_{\mathrm{AIF}}(v, 1-u, w, 1-w)) \ge \max(h_{\mathrm{AIF}}(u, 1-u, v, 1-v), h_{\mathrm{AIF}}(v, 1-v, w, 1-w)) = \max(h(u,v), h(v,w))$.

Then, since h satisfies loc.1–loc.3, applying [23, Proposition 3.4], we conclude that D is a local divergence. ■

On the other hand, let us investigate what happens with respect to locality under the conditions of Proposition 5.2.

Proposition 5.4: Under the conditions of Proposition 5.2, if D_1 and D_2 are two local divergences, with associated functions h_1 and h_2, respectively, then the AIF-divergence D_{AIF}, defined by (13), is local if and only if f can be expressed by $f(u,v) = \alpha u + \beta v$, for some $\alpha, \beta \geq 0$. In such a case, the function h_{AIF} associated with D_{AIF} is given by $h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2) = \alpha h_1(u_1,v_1) + \beta h_2(u_2,v_2)$.

Proof: Assume that both D_1 and D_2 are two local divergences with associated functions h_1 and h_2, that satisfy properties from loc.1 to loc.3. From Proposition 5.2, D_{AIF} is defined by $D_{\mathrm{AIF}}(A,B) = f(\sum_{i=1}^n h_1(\mu_A(x_i),\mu_B(x_i)), \sum_{i=1}^n h_2(\nu_A(x_i),\nu_B(x_i)))$.

Assume, on one hand, that $f(u,v) = \alpha u + \beta v$. Then, $D_{\mathrm{AIF}}(A,B) = \alpha \sum_{i=1}^n h_1(\mu_A(x_i),\mu_B(x_i)) + \beta \sum_{i=1}^n h_2(\nu_A(x_i), \nu_B(x_i)) = \sum_{i=1}^n \alpha h_1(\mu_A(x_i),\mu_B(x_i)) + \beta h_2(\nu_A(x_i),\nu_B(x_i))$.

Thus, D_{AIF} is local and $h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2) = \alpha h_1(u_1, v_1) + \beta h_2(u_2,v_2)$. In addition, h_{AIF} satisfies properties AIF-loc.1–AIF-loc.5.

On the other hand, if D_{AIF} is a local AIF-divergence with associated function h_{AIF}, then $f(\sum_{i=1}^n h_1(\mu_A(x_i),\mu_B(x_i)), \sum_{i=1}^n h_2(\nu_A(x_i),\nu_B(x_i))) = \sum_{i=1}^n h_{\mathrm{AIF}}(\mu_A(x_i),\nu_A(x_i),\mu_B(x_i),\nu_B(x_i))$.

However, the previous equality holds only if f is a linear function, that is, if $f(u,v) = \alpha u + \beta v$. In addition, both coefficients α and β must be nonnegative since the AIF-divergence is a nonnegative measure. ■

Next example shows how these results can be applied.

Example 5.5: Let us consider the local AIF-divergence D_{HK} of Hong and Kim defined in (7). On one hand, if we apply Proposition 5.1, we obtain the following divergence: For every $A, B \in FS(X)$, $D(A,B) = D_{\mathrm{HK}}(A,B) = \frac{1}{2n}\sum_{i=1}^n |A(x_i) - B(x_i)| + |(1-A(x_i)) - (1-B(x_i))| = \frac{1}{n}\sum_{i=1}^n |A(x_i) - B(x_i)|$.

This divergence is known as the Hamming distance for fuzzy sets (see [28]), and it is usually denoted by l_{FS}. Moreover, as the Hong and Kim AIF-divergence satisfies the local property, applying Proposition 5.3, the Hamming distance for fuzzy sets is also a local divergence.

On the other hand, if we consider the function $f(u,v) = \frac{u+v}{2}$ and we apply Proposition 5.2 to the Hamming distance for fuzzy sets, we obtain the following AIF-divergence: $D_{\mathrm{AIF}}(A,B) = f(l_{FS}(\mu_A,\mu_B), l_{FS}(\nu_A,\nu_B)) = \frac{l_{FS}(\mu_A,\mu_B) + l_{FS}(\nu_A,\nu_B)}{2} = \frac{1}{2}(\frac{1}{n}\sum_{i=1}^n |\mu_A(x_i) - \mu_B(x_i)| + \frac{1}{n}\sum_{i=1}^n |\nu_A(x_i) - \nu_B(x_i)|) = \frac{1}{2n}\sum_{i=1}^n (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) = D_{\mathrm{HK}}(A,B)$.

That is, we obtain the original AIF-divergence, which is known to be local. However, we could also derive the locality from the fact that D is a local divergence for fuzzy sets and f is a linear function with positive parameters.

On the other hand, if we consider the function $f^*(u,v) = (u^2+v^2)^{\frac{1}{2}}$ and we apply Proposition 5.2, we obtain the following AIF-divergence: $D^*_{\mathrm{AIF}}(A,B) = f(D(\mu_A,\mu_B), D(\nu_A,\nu_B)) = (D(\mu_A,\mu_B)^2 + D(\nu_A,\nu_B)^2)^{\frac{1}{2}} = ((\sum_{i=1}^n |\mu_A(x_i) - \mu_B(x_i)|)^2 + (\sum_{i=1}^n |\nu_A(x_i) - \nu_B(x_i)|)^2)^{\frac{1}{2}}$.

Although the Hamming distance for fuzzy sets is a local divergence, since f is not a linear function, Proposition 5.2 assures that the obtained AIF-divergence D_{AIF} does not satisfy the local property.

The previous example shows that there are situations in which from an AIF-divergence D_{AIF}, it is possible to define another AIF-divergence, using Proposition 5.1 and Proposition 5.2, that coincides with the original. Obviously, it does not happen always, as we saw in the previous example. In the rest of this section, we study when this commutativity holds.

Theorem 5.6: Consider an AIF-divergence D_{AIF} and the divergence D defined by (12). Let D^*_{AIF} be the AIF-divergence defined through the function f as in (13) taking $D_1 = D_2 = D$, that is, $D^*_{\mathrm{AIF}}(A,B) = f(D(\mu_A,\mu_B), D(\nu_A,\nu_B))$ for every $A, B \in AIFS(X)$.

Then, $D_{\mathrm{AIF}} = D^*_{\mathrm{AIF}}$ if and only if for every $A, B \in AIFS(X)$, the following equality holds: $D_{\mathrm{AIF}}(A,B) = f(D_{\mathrm{AIF}}(\mu_A, \mu_B), D_{\mathrm{AIF}}(\nu_A,\nu_B))$.

In fact, if D_{AIF} is local, with associated function h_{AIF}, $D_{\mathrm{AIF}} = D^*_{\mathrm{AIF}}$ if and only if $f(u,v) = \alpha u + \beta v$, for some $\alpha, \beta \geq 0$, and $h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2) = \alpha h_{\mathrm{AIF}}(u_1, 1-u_1, v_1, 1-v_1) + \beta h_{\mathrm{AIF}}(u_2, 1-u_2, v_2, 1-v_2)$.

Proof: To prove this result, it suffices to compute the expression of the AIF-divergence D^*_{AIF}, that is, $D^*_{\mathrm{AIF}}(A,B) = f(D(\mu_A,\mu_B), D(\nu_A,\nu_B)) = f(D_{\mathrm{AIF}}(\mu_A,\mu_B), D_{\mathrm{AIF}}(\nu_A,\nu_B))$.

Take into account that μ_A and μ_B are the AIF-sets whose nonmembership functions are given by $1-\mu_A$ and $1-\mu_B$, respectively.

Assume now that D_{AIF} is local, with associated function h_{AIF}. In such a case, Proposition 5.3 assures that D is local, with associated function $h(u,v) = h_{\mathrm{AIF}}(u, 1-u, v, 1-v)$. Developing the expression of $D^*_{\mathrm{AIF}}(A,B)$, we obtain that $D^*_{\mathrm{AIF}}(A,B) = f(D(\mu_A,\mu_B), D(\nu_A,\nu_B)) = f(\sum_{i=1}^n h_{\mathrm{AIF}}(\mu_A(x_i), 1-\mu_A(x_i), \mu_B(x_i), 1-\mu_B(x_i)), \sum_{i=1}^n h_{\mathrm{AIF}}(\nu_A(x_i), 1-\nu_A(x_i), \nu_B(x_i), 1-\nu_B(x_i)))$.

By Proposition 5.4, we know that D^*_{AIF} is local if and only if $f(u,v) = \alpha u + \beta v$, for some $\alpha, \beta \geq 0$. In such a case, $D^*_{\mathrm{AIF}}(A,B) = \alpha \sum_{i=1}^n h_{\mathrm{AIF}}(\mu_A(x_i), 1-\mu_A(x_i), \mu_B(x_i), 1-\mu_B(x_i)) + \beta \sum_{i=1}^n h_{\mathrm{AIF}}(\nu_A(x_i), 1-\nu_A(x_i), \nu_B(x_i), 1-\nu_B(x_i))$.

Thus, $D_{\mathrm{AIF}} = D^*_{\mathrm{AIF}}$ if and only if $\sum_{i=1}^n h_{\mathrm{AIF}}(\mu_A(x_i), \nu_A(x_i), \mu_B(x_i), \nu_B(x_i)) = \alpha \sum_{i=1}^n h_{\mathrm{AIF}}(\mu_A(x_i), 1-\mu_A(x_i), \mu_B(x_i), 1-\mu_B(x_i)) + \beta \sum_{i=1}^n h_{\mathrm{AIF}}(\nu_A(x_i), 1-\nu_A(x_i), \nu_B(x_i), 1-\nu_B(x_i))$.

Consequently, $D_{\mathrm{AIF}} = D^*_{\mathrm{AIF}}$ if and only if $f(u,v) = \alpha u + \beta v$ and $h_{\mathrm{AIF}}(u_1,u_2,v_1,v_2) = \alpha h_{\mathrm{AIF}}(u_1, 1-u_1, v_1, 1-v_1) + \beta h_{\mathrm{AIF}}(u_2, 1-u_2, v_2, 1-v_2)$ for every $(u_1,u_2), (v_1,v_2) \in \mathcal{D} = \{(u,v) \in [0,1]^2 : u+v \leq 1\}$. ■

Conversely, we now characterize the situations under which a divergence for fuzzy sets coincide with the divergence obtained by applying Propositions 5.1 and 5.2.

Theorem 5.7: Let D be a divergence for fuzzy sets, and let D_{AIF} be the AIF-divergence derived from Proposition 5.2 using the function f. Denote by D^* the divergence derived from D_{AIF} by using Proposition 5.1. Then, $D = D^*$ if and only if $f(u,v) = u$ for every $(u,v) \in \mathcal{D}$.

Proof: Let us compute the expression of D^*: $D^*(A,B) = D_{\mathrm{AIF}}(A,B) = f(D(A,B), D(A^c,B^c))$. Thus, $D(A,B) = D^*(A,B)$ if and only if $f(u,v) = u$. ■

Note that if D is a local divergence, the obtained result is also local, since $f(u,v) = u$ is a linear function.

Example 5.8: Consider the Hamming distance for fuzzy sets $l_{FS}(A,B)=\sum_{i=1}^{n}|A(x_i)-B(x_i)|$, for every $A,B\in FS(X)$. It is obvious that this divergence is local with associated function $h(u,v)=|u-v|$. Applying Proposition 5.2, we can build an AIF-divergence: $D_{\text{AIF}}(A,B)=f(\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|, \sum_{i=1}^{n}|\nu_A(x_i)-\nu_B(x_i)|)$.

In addition, from Proposition 5.4, D_{AIF} is local if and only if $f(u,v)=\alpha u+\beta v$ for some $\alpha,\beta\geq 0$. Using Proposition 5.1, it is possible to build another divergence: $D^*(A,B)= f(\sum_{i=1}^{n}|A(x_i)-B(x_i)|,\sum_{i=1}^{n}|A(x_i)-B(x_i)|)$.

Then, $D^*(A,B)=D(A,B)$ if and only if $f(u,u)=u$. In particular, both divergences are the same if $f(u,v)=u$, as Theorem 5.7 assures.

Consider now the AIF-divergence D_{HK} defined by Hong and Kim in (7). Applying Proposition 5.1, we can build a divergence for fuzzy sets: $D(A,B)=D_{\text{HK}}(A,B)=\frac{1}{n}\sum_{i=1}^{n}|A(x_i)-B(x_i)|=l_{FS}(A,B)$.

If we now apply Proposition 5.2, we can build another AIF-divergence given by $D_{\text{AIF}}(A,B)=f(D(\mu_A,\mu_B),D(\nu_A,\nu_B)) =f(\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|,\frac{1}{n}\sum_{i=1}^{n}|\nu_A(x_i)-\nu_B(x_i)|)$.

Thus, we conclude that $D_{\text{AIF}}(A,B)=D_{\text{HK}}(A,B)$ if and only if $f(x,y)=\frac{x+y}{2}$.

Our final result considers Theorems 5.6 and 5.7 simultaneously to see under which conditions, the methods described in Propositions 5.1 and 5.2 are commutative.

Theorem 5.9: Let D be a divergence for fuzzy sets. Using Proposition 5.2, we can define an AIF-divergence D_{AIF}. Using Proposition 5.1, we define another divergence D^*, and finally, using Proposition 5.2, once more, we define another AIF-divergence D^*_{AIF}. Then, $D=D^*$ and $D_{\text{AIF}}=D^*_{\text{AIF}}$ if and only if $f(u,v)=u$ and $D_{\text{AIF}}(A,B)=D(\mu_A,\mu_B)$ for every $A,B\in AIFS(X)$.

In such conditions, if D is local with associated function h, D_{AIF} is also local with associated function $h_{\text{AIF}}(u_1,u_2,v_1,v_2) =h(u_1,v_1)$.

Proof: On one hand, from Theorem 5.7, we know that $D= D^*$ if and only if $f(u,v)=u$. Moreover, from Theorem 5.6, $D_{\text{AIF}}=D^*_{\text{AIF}}$ if and only if $D_{\text{AIF}}(A,B)=f(D_{\text{AIF}}(\mu_A,\mu_B), D_{\text{AIF}}(\nu_A,\nu_B))$. As $f(u,v)=u$, $D_{\text{AIF}}=D^*_{\text{AIF}}$ if and only if $D_{\text{AIF}}(A,B)=D_{\text{AIF}}(\mu_A,\mu_B)=D(\mu_A,\mu_B)$. In such a case, if D is local with associated function h, D_{AIF} is also local since f is linear (Proposition 5.2), and it holds that: $D_{\text{AIF}}(A,B)= D(\mu_A,\mu_B)=\sum_{i=1}^{n}h(\mu_A(x_i),\mu_B(x_i))$. Thus, $h_{\text{AIF}}(u_1,u_2, v_1,v_2)=h(u_1,v_1)$. ■

VI. Applications

Up to now, we have investigated local AIF-divergences and their main properties. However, local AIF-divergences are not only interesting from a theoretical point of view, but they also have several applications. In this section, we show how they can be applied in two different fields: multiple attribute decision making and pattern recognition.

A. Application to Pattern Recognition

One interesting area of application of comparison measures for AIF-sets is in pattern recognition [16], [17], [19]. Let us consider a universe $X=\{x_1,\ldots,x_n\}$, and assume the patterns $A_1,\ldots,A_m$, are represented by AIF-sets. Here, x_i's are attributes and A_i's can be viewed as prototypes. Then, $A_j= \{(x_i,\mu_{A_j}(x_i),\nu_{A_j}(x_i):i=1,\ldots,n\}$, for $j=1,\ldots,n$.

If B is a sample that is also represented by an AIF-set, and we want to classify it into one of the patterns, we can measure the difference between B and A_i as $D_{\text{AIF}}(A_1,B),\ldots,D_{\text{AIF}}(A_m, B)$, where D_{AIF} is an AIF-divergence. Finally, we associate B to the pattern A_j whenever $D_{\text{AIF}}(A_j,B)=\min_{i=1,\ldots,m}(D_{\text{AIF}}(A_i,B))$, i.e., we classify B into the pattern from which it differs the least.

Example 6.1 (see[19, Sec. 4]): Let us consider an universe with three elements, $X=\{x_1,x_2,x_3\}$, and the following three patterns: $A_1=\{(x_1,0.1,0.1),(x_2,0.5,0.4),(x_3,0.1,0.9)\}$. $A_2 =\{(x_1,0.5,0.5),(x_2,0.7,0.3),(x_3,0,0.8)\}$ and $A_3=\{(x_1, 0.7,0.2),(x_2,0.1,0.8),(x_3,0.4,0.4)\}$.

Assume that a sample $B=\{(x_1,0.4,0.4),(x_2,0.6,0.2), (x_3,0,0.8)\}$ is given, and let us consider the Hamming and the Hausdorff distances for AIF-sets. We obtain the following results:

	A_1	A_2	A_3
$l_{\text{IFS}}(A_i,B)$	1	0.4	1.3
$d_{\text{H}}(A_i,B)$	0.6	0.2	1.3

Thus, both AIF-divergences classify B into the pattern A_2, because $l_{\text{IFS}}(A_2,B)\leq l_{\text{IFS}}(A_1,B),l_{\text{IFS}}(A_3,B)$ and $d_{\text{H}}(A_2, B)\leq d_{\text{H}}(A_1,B),d_{\text{H}}(A_3,B)$.

This is a particular example in which no assumption is made over the elements on X. This is equivalent to assuming that attributes are equally weighted: The weight α_i of the attribute x_i is $\frac{1}{n}$, where n is the number of attributes. However, in the framework of pattern recognition, it is possible to have different α_i's for different attributes. This problem cannot been tackled using the approaches in [16], [17], [19], and [31]. However, our approach can be adapted using the local property. For this aim, let us consider a local AIF-divergence D_{AIF}, and for every attribute x_i, using (2), we compute the following difference $D_{\text{AIF}}(A_j,B)-D_{\text{AIF}}(A_j\cup\{x_i\},B\cup\{x_i\})=h_{\text{AIF}}(\mu_{A_j}(x_i),\nu_{A_j}(x_i),\mu_B(x_i),\nu_B(x_i))$. Then, for every $j\in\{1,\ldots,m\}$, we have that $d(A_j,B)=\sum_{i=1}^{n}\alpha_i(D_{\text{AIF}}(A_j,B)-D_{\text{AIF}}(A_j\cup\{x_i\},B\cup\{x_i\}))=\sum_{i=1}^{n}\alpha_i h_{\text{AIF}}(\mu_{A_j}(x_i),\nu_{A_j}(x_i),\mu_B(x_i),\nu_B(x_i))$, where α_i are the weights of the attributes ($\alpha_i>0$ and $\sum_{i=1}^{n}\alpha_i=1$). This function $d(A_j,B)$ computes the differences between A_j and B in each attribute and, then, aggregates them according to the respective weights. Finally, we classify the sample B into the pattern A_j if $d(A_j,B)= \min_{i=1,\ldots,m}(d(A_i,B))$.

Note that the key of this approach is the fact that the AIF-divergence can be computed pointwise, and therefore, it is possible to add the weight of each attribute. In general, if we follow the usual procedures in [16], [17], [19], and [31], this step cannot be done. To show how to apply it, we consider a slightly modified version of [31, Example 4.2], in which we shall assume that the attributes have different weights.

Example 6.2 (see[31, Example 4.2]): Consider five mineral fields where each field is featured by the content of six minerals

TABLE I
SIX KINDS OF MATERIALS ARE REPRESENTED BY AIF-SETS

	x_1	x_2	x_3	x_4	x_5	x_6
$\mu_{A_1}(x_i)$	0.739	0.033	0.188	0.492	0.020	0.739
$\nu_{A_1}(x_i)$	0.125	0.818	0.626	0.358	0.628	0.125
$\mu_{A_2}(x_i)$	0.124	0.030	0.048	0.136	0.019	0.393
$\nu_{A_2}(x_i)$	0.665	0.825	0.800	0.648	0.823	0.653
$\mu_{A_3}(x_i)$	0.449	0.662	1.000	1.000	1.000	1.000
$\nu_{A_3}(x_i)$	0.387	0.298	0.000	0.000	0.000	0.000
$\mu_{A_4}(x_i)$	0.280	0.521	0.470	0.295	0.188	0.735
$\nu_{A_4}(x_i)$	0.715	0.368	0.423	0.658	0.806	0.118
$\mu_{A_5}(x_i)$	0.326	1.000	0.182	0.156	0.049	0.675
$\nu_{A_5}(x_i)$	0.452	0.000	0.725	0.765	0.896	0.263
$\mu_B(x_i)$	0.629	0.524	0.210	0.218	0.069	0.658
$\nu_B(x_i)$	0.303	0.356	0.689	0.753	0.876	0.256

and each field contains one kind of typical hybrid mineral. The five kinds of typical hybrid minerals are represented by AIF-sets A_1, A_2, A_3, A_4, and A_5 in $X = \{x_1, \ldots, x_6\}$, respectively. Assume that we are given another kind of hybrid mineral B, and that we want to classify it into one of the aforementioned mineral fields. Assume that the AIF-sets A_i and B are defined in Table I, and that our experts have established the following weight vector on X: $\alpha = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right\}$. Let us use our method to classify B. If we consider the Hamming distance for AIF-sets as local AIF-divergence, we obtain that the values of $l_{\mathrm{IFS}}(A_j, B) - l_{\mathrm{IFS}}(A_j \cup \{x_i\}, B \cup \{x_i\})$ are

	x_1	x_2	x_3	x_4	x_5	x_6
$j=1$	0.178	0.491	0.085	0.395	0.297	0.131
$j=2$	0.505	0.494	0.162	0.187	0.103	0.397
$j=3$	0.180	0.138	0.790	0.782	0.931	0.342
$j=4$	0.412	0.012	0.266	0.095	0.119	0.138
$j=5$	0.303	0.476	0.036	0.062	0.020	0.024

whence $d(A_1, B) = \frac{1}{4}0.178 + \frac{1}{4}0.491 + \frac{1}{8}0.085 + \frac{1}{8}0.395 + \frac{1}{8}0.297 + \frac{1}{8}0.131 = 0.2808$, $d(A_2, B) = \frac{1}{4}0.505 + \frac{1}{4}0.494 + \frac{1}{8}0.162 + \frac{1}{8}0.187 + \frac{1}{8}0.103 + \frac{1}{8}0.397 = 0.3559$, $d(A_3, B) = \frac{1}{4}0.180 + \frac{1}{4}0.138 + \frac{1}{8}0.790 + \frac{1}{8}0.782 + \frac{1}{8}0.931 + \frac{1}{8}0.342 = 0.4351$, $d(A_4, B) = \frac{1}{4}0.412 + \frac{1}{4}0.012 + \frac{1}{8}0.266 + \frac{1}{8}0.095 + \frac{1}{8}0.119 + \frac{1}{8}0.138 = 0.1833$, and $d(A_5, B) = \frac{1}{4}0.303 + \frac{1}{4}0.476 + \frac{1}{8}0.036 + \frac{1}{8}0.062 + \frac{1}{8}0.020 + \frac{1}{8}0.024 = 0.2125$.

Thus, we classify B into the hybrid mineral A_4.

If we repeat the process with local AIF-divergence d_{H}, we obtain the following values of $d_{\mathrm{H}}(A_j, B) - d_{\mathrm{H}}(A_j \cup \{x_i\}, B \cup \{x_i\})$:

	x_1	x_2	x_3	x_4	x_5	x_6
$j=1$	0.178	0.491	0.063	0.395	0.248	0.131
$j=1$	0.505	0.494	0.162	0.105	0.053	0.397
$j=3$	0.180	0.138	0.790	0.782	0.931	0.342
$j=4$	0.412	0.012	0.266	0.095	0.119	0.138
$j=5$	0.303	0.476	0.036	0.062	0.020	0.017

Then, $d(A_1, B) = \frac{1}{4}0.178 + \frac{1}{4}0.491 + \frac{1}{8}0.063 + \frac{1}{8}0.395 + \frac{1}{8}0.248 + \frac{1}{8}0.131 = 0.2719$, $d(A_2, B) = \frac{1}{4}0.505 + \frac{1}{4}0.494 + \frac{1}{8}0.162 + \frac{1}{8}0.105 + \frac{1}{8}0.053 + \frac{1}{8}0.397 = 0.3394$, $d(A_3, B) = \frac{1}{4}0.180 + \frac{1}{4}0.138 + \frac{1}{8}0.790 + \frac{1}{8}0.782 + \frac{1}{8}0.931 + \frac{1}{8}0.342 = 0.4351$, $d(A_4, B) = \frac{1}{4}0.412 + \frac{1}{4}0.012 + \frac{1}{8}0.266 + \frac{1}{8}0.095 + \frac{1}{8}0.119 + \frac{1}{8}0.138 = 0.1833$, and $d(A_5, B) = \frac{1}{4}0.303 + \frac{1}{4}0.476 + \frac{1}{8}0.036 + \frac{1}{8}0.062 + \frac{1}{8}0.020 + \frac{1}{8}0.017 = 0.2116$, and we conclude that we also should classify B into the hybrid mineral A_4.

We consider that our approach fits better than previous approaches in pattern recognition problems where different weights are associated with different attributes. As we have seen, the local property plays a key role because it allows one to decompose the computation of the difference between each attribute and the object to a pointwise computation.

B. Application to Decision Making

In [32], Xu showed how measures of similarity for AIF-sets can be applied within multiple attribute decision making. Let us overview the main aspects of this application.

We use the following notation: Let $A = \{A_1, \ldots, A_m\}$ denote a set of m alternatives, let $C = \{C_1, \ldots, C_n\}$ be a set of attributes, and let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be its associated weight vector (i.e., they satisfy that $\alpha_i \geq 0$ for every $i = 1, \ldots, n$ and that $\alpha_1 + \ldots + \alpha_n = 1$).

Every alternative A_i can be represented by means of an AIF-set $A_i = \{(C_j, \mu_{A_i}(C_j), \nu_{A_i}(C_j) : j = 1, \ldots, n\}$. Thus, $\mu_{A_i}(C_j)$ and $\nu_{A_i}(C_j)$ stand for the degree in which alternative A_i agrees and does not agree with characteristic C_j, respectively.

Xu [32] defined the AIF-sets A^+ and A^- in the following way: $A^+ = \{(C_j, \mu_{A^+}(C_j), \nu_{A^+}(C_j)) : j = 1, \ldots, n\}$ and $A^- = \{(C_j, \mu_{A^-}(C_j), \nu_{A^-}(C_j)) : j = 1, \ldots, n\}$, where $\mu_{A^+}(C_j) = \max_{i=1,\ldots,m}(\mu_{A_i}(C_j))$, $\nu_{A^+}(C_j) = \min_{i=1,\ldots,m}(\nu_{A_i}(C_j))$, $\mu_{A^-}(C_j) = \min_{i=1,\ldots,m}(\mu_{A_i}(C_j))$ and $\nu_{A^-}(C_j) = \max_{i=1,\ldots,m}(\nu_{A_i}(C_j))$, that is, $A^+ = \bigcup_{i=1}^m A_i$ and $A^- = \bigcap_{i=1}^m A_i$.

These AIF-sets can be interpreted as the "optimal" and the "least optimal" alternatives. Therefore, the preferred alternative in A would be the one that is simultaneously most similar to A^+ and most different to A^-.

In order to measure how different is A_i from both A^+ and A^-, Xu considered one of the following functions: $D_1(A, B) = [\sum_{j=1}^n \alpha_j(|\mu_A(C_j) - \mu_B(C_j)|^\beta + |\nu_A(C_j) - \nu_B(C_j)|^\beta + |\pi_A(C_j) - \pi_{\mathrm{B}}(C_j)|^\beta)]^{\frac{1}{\beta}}$, $D_2(A, B) = ((\sum_{j=1}^n \alpha_j(|\mu_A(C_j) - \mu_B(C_j)|^\beta + |\nu_A(C_j) - \nu_B(C_j)|^\beta) + |\pi_A(C_j) - \pi_{\mathrm{B}}(C_j)|^\beta)/x(\sum_{j=1}^n \alpha_j(|\mu_A(C_j) + \mu_B(C_j)|^\beta + |\nu_A(C_j) + \nu_B(C_j)|^\beta) + |\pi_A(C_j) + \pi_{\mathrm{B}}(C_j)|^\beta))^{\frac{1}{\beta}}$, $D_3(A, B) = (\sum_{j=1}^n \alpha_j(\min(\mu_A(C_j), \mu_B(C_j)) + \min(\nu_A(C_j), \nu_B(C_j)) + \min(\pi_A(C_j), \pi_{\mathrm{B}}(C_j))))/(\sum_{j=1}^n \alpha_j(\max(\mu_A(C_j), \mu_B(C_j)) + \max(\nu_A(C_j), \nu_B(C_j)) + \max(\pi_A(C_j), \pi_{\mathrm{B}}(C_j))))$, and $D_4(A, B) = (\sum_{j=1}^n \alpha_j(\mu_A(C_j)\mu_B(C_j) + \nu_A(C_j)\nu_B(C_j) + \pi_A(C_j)\pi_{\mathrm{B}}(C_j)))/(\max(\sum_{j=1}^n \alpha_j(\mu_A^2(C_j) + \nu_A^2(C_j) + \pi_A^2(C_j)), \sum_{j=1}^n \alpha_j(\mu_B^2(C_j) + \nu_B^2(C_j) + \pi_{\mathrm{B}}^2(C_j))))$.

Besides, Xu consider the quotient: $d_i = \frac{D_j(A^+, A_i)}{D_j(A^+, A_i) + D(A^-, A_i)}$ for $j = 1, 2, 3, 4$. Then, the greater the value d_i, the worst is the alternative A_i.

This was the approach followed in [32]. However, we think that his proposal can be improved in two directions.

1) First of all, none of the functions D_j are AIF-divergences, except for the trivial case of D_1 with $\beta = 1$. We have already argued in [26] that from our point of view, AIF-divergences are the adequate measures of comparison of AIF-sets because they require more restrictive conditions, and therefore, they can avoid counterintuitive measures.
2) Second, since the measure of comparison chosen in [32] cannot be computed pointwise, the weights cannot be put into the measure.

Taking into account these two comments, we next propose a modification of the above method using local AIF-divergences. Let us consider a local AIF-divergence D_{AIF} so that for every pair of AIF-sets A and B, $D_{\text{AIF}}(A, B)$ can be expressed by $D_{\text{AIF}}(A, B) = \sum_{i=1}^{n} h_{\text{AIF}}(\mu_A(C_i), \nu_A(C_i), \mu_B(C_i), \nu_B(C_i))$.

We consider the AIF-set A_i, which represents the ith alternative, and for every $j \in \{1, \ldots, n\}$, we compute the following: $D_{\text{AIF}}(A^+, A_i) - D_{\text{AIF}}(A^+ \cup \{C_j\}, A_i \cup \{C_j\}) = h_{\text{AIF}}(\mu_{A^+}(C_j), \nu_{A^+}(C_j), \mu_{A_i}(C_j), \nu_{A_i}(C_j))$.

This quantity measures how different are A^+ and A_i in the element C_j. Then, we can compute the difference between A_i and A^+: $d(A_i, A^+) = \sum_{j=1}^{n} \alpha_j h_{\text{AIF}}(\mu_{A^+}(C_j), \nu_{A^+}(C_j), \mu_{A_i}(C_j), \nu_{A_i}(C_j))$. This way, $d(A_i, A^+)$ measures how much difference there is between A_i and the optimal set A^+.

Similarly, we can compute the difference between A_i and A^-: $d(A_i, A^-) = \sum_{j=1}^{n} \alpha_j h_{\text{AIF}}(\mu_{A^-}(C_j), \nu_{A^-}(C_j), \mu_{A_i}(C_j), \nu_{A_i}(C_j))$. In a similar way, $d(A_i, A^-)$ measures how A_i differs from the least optimal A^-.

Thus, if we consider a map $\psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ that is nonincreasing in the first component and nondecreasing on the second one, we obtain the following value a_i for alternative A_i: $a_i = \psi(d(A_i, A^+), d(A_i, A^-))$. Thus, the greater the value of a_i, the more preferred is the alternative A_i.

Obviously, we can choose the function ψ depending on which part we are more interested: the difference between A_i and the optimum A^+ or the difference between A_i and the least optimum A^-. The following examples illustrate this fact.

Example 6.3 (see[32, Sec. 4]): A city is planning to build a library, and the city commissioner has to determine the air-conditioning system to be installed in the library. The builder offers the commissioner five feasible alternatives $A_i; i = 1, \ldots, 5$, which might be adapted to the physical structure of the library. Suppose that three attributes C_1 (economic), C_2 (functional), and C_3 (operational) are taken into consideration in the installation problem, and that the weight vector of the attributes C_j is $\alpha = (0.3, 0.5, 0.2)$. Assume, moreover, that the characteristics of the alternatives A_i are represented by the following AIF-sets: $A_1 = \{(C_1, 0.2, 0.4), (C_2, 0.7, 0.1), (C_3, 0.6, 0.3)\}$, $A_2 = \{(C_1, 0.4, 0.2), (C_2, 0.5, 0.2), (C_3, 0.8, 0.1)\}$, $A_3 = \{(C_1, 0.5, 0.4), (C_2, 0.6, 0.2), (C_3, 0.9, 0)\}$, $A_4 = \{(C_1, 0.3, 0.5), (C_2, 0.8, 0.1), (C_3, 0.7, 0.2)\}$, and $A_5 = \{(C_1, 0.8, 0.2), (C_2, 0.7, 0), (C_3, 0.1, 0.6))\}$.

For these AIF-sets, the corresponding A^+ and A^- are given by $A^+ = \{(C_1, 0.8, 0.2), (C_2, 0.8, 0), (C_3, 0.9, 0)\}$ and $A^- = \{(C_1, 0.2, 0.5), (C_2, 0.5, 0.2), (C_3, 0.1, 0.6)\}$. Then, if we consider the Hamming distance for AIF-sets, we obtain the following values for $l_{\text{IFS}}(A_i, A^+) - l_{\text{IFS}}(A_i \cup \{C_j\}, A^+ \cup \{C_j\})$ and $l_{\text{IFS}}(A_i, A^-) - l_{\text{IFS}}(A_i \cup \{C_j\}, A^- \cup \{C_j\})$

	C_1	C_2	C_3
$i = 1,\ A^+$	1.2	0.2	0.6
$i = 1,\ A^-$	0.2	0.4	1
$i = 2,\ A^+$	0.8	0.6	0.2
$i = 2,\ A^-$	0.6	0	1.4
$i = 3,\ A^+$	0.6	0.4	0
$i = 3,\ A^-$	0.6	0.2	1.6
$i = 4,\ A^+$	1	0.2	0.4
$i = 4,\ A^-$	0.2	0.6	1.2
$i = 5,\ A^+$	0	0.2	1.6
$i = 5,\ A^-$	1.2	0.4	0

Thus

	A_1	A_2	A_3	A_4	A_5
$d(A_i,\ A^+)$	0.58	0.58	0.38	0.48	0.42
$d(A_i,\ A^-)$	0.46	0.46	0.60	0.60	0.56.

Assume that we want to choose the alternative that is, at the same time, more similar to the optimum A^+ and less similar to the least optimum A^-. In such a case, we can consider the function ψ given by $\psi(x, y) = \frac{1}{2}\left(\frac{1}{x} + y\right)$ assuming $x > 0$. We can see that this function takes into account the difference between A_i and A^+ and between A_i and A^-. We obtain the following results:

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = \frac{1}{2}\left(\frac{1}{0.58} + 0.46\right) = 1.09$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = \frac{1}{2}\left(\frac{1}{0.58} + 0.46\right) = 1.09$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = \frac{1}{2}\left(\frac{1}{0.38} + 0.6\right) = 1.62$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = \frac{1}{2}\left(\frac{1}{0.48} + 0.6\right) = 1.34$$

and

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = \frac{1}{2}\left(\frac{1}{0.42} + 0.56\right) = 1.47.$$

Since $a_3 > a_5 > a_4 > a_1 = a_2$, we decide in favor of A_3.

Now, let us assume that we decide to choose the alternative that is more similar to the optimum A^+, regardless of the difference from A^-. In that case, we may consider $\psi(x, y) = \frac{1}{x}$. This function only depends on the difference between A_i and the optimum A^+. We obtain the following result:

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = \frac{1}{d(A_1, A^+)} = \frac{1}{0.58}$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = \frac{1}{d(A_2, A^+)} = \frac{1}{0.58}$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = \frac{1}{d(A_3, A^+)} = \frac{1}{0.38}$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = \frac{1}{d(A_4, A^+)} = \frac{1}{0.48} \text{ and}$$

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = \frac{1}{d(A_5, A^+)} = \frac{1}{0.42}.$$

Thus, $A_3 \succ A_5 \succ A_4 \succ A_1 \sim A_2$, and as a consequence, the best alternative is A_3.

Finally, let us assume that we are interested in the alternative that differs most from the worst alternative A^-. In such a situation, we should consider $\psi(x, y) = y$. This function only depends on the difference between A_i and A^-. We obtain the following results:

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = d(A_1, A^-) = 0.46$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = d(A_2, A^-) = 0.46$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = d(A_3, A^-) = 0.6$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = d(A_4, A^-) = 0.6$$

and

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = d(A_5, A^-) = 0.56.$$

Thus, we obtain that $A_3 \sim A_4 \succ A_5 \succ A_1 \sim A_2$. We conclude that in this case A_3 and A_4 are the preferred alternatives. ■

Example VI.4: Consider the previous example, but now with the Hausdorff distance for AIF-sets. Using the same AIF-sets, we obtain that the values of $d_{\mathrm{H}}(A_i, A^+) - d_{\mathrm{H}}(A_i \cup \{C_j\}, A^+ \cup \{C_j\})$ and $d_{\mathrm{H}}(A_i, A^-) - d_{\mathrm{H}}(A_i \cup \{C_j\}, A^- \cup \{C_j\})$ are

	C_1	C_2	C_3
$i = 1,\ A^+$	0.6	0.1	0.3
$i = 1,\ A^-$	0.3	0.2	0.5
$i = 2,\ A^+$	0.4	0.3	0.1
$i = 2,\ A^-$	0.3	0	0.7
$i = 3,\ A^+$	0.3	0.2	0
$i = 3,\ A^-$	0.3	0.1	0.8
$i = 4,\ A^+$	0.5	0.1	0.2
$i = 4,\ A^-$	0.3	0.3	0.6
$i = 5,\ A^+$	0	0.1	0.8
$i = 5,\ A^-$	0.6	0.2	0

Then

	A_1	A_2	A_3	A_4	A_5
$d(A_i, A^+)$	0.29	0.3	0.19	0.26	0.29
$d(A_i, A^-)$	0.34	0.3	0.38	0.42	0.28.

As before, we first look for the alternative that is, at the same time, most similar to the optimum A^+ and least similar to the least optimum A^- with the function $\psi(x, y) = \frac{1}{2}\left(\frac{1}{x} + y\right)$. It holds that

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = \frac{1}{2}\left(\frac{1}{0.29} + 0.34\right) = 3.79$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = \frac{1}{2}\left(\frac{1}{0.3} + 0.3\right) = 3.63$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = \frac{1}{2}\left(\frac{1}{0.19} + 0.38\right) = 5.64$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = \frac{1}{2}\left(\frac{1}{0.26} + 0.42\right) = 4.27$$

and

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = \frac{1}{2}\left(\frac{1}{0.29} + 0.28\right) = 3.72.$$

Then, $A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$, and again, A_3 is the preferred alternative.

Now, we want to consider the alternative that is most similar to the optimal A^+. Then, a possible function ψ is $\psi(x, y) = \frac{1}{x}$. In such a case,

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = \frac{1}{d(A_1, A^+)} = \frac{1}{0.29}$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = \frac{1}{d(A_2, A^+)} = \frac{1}{0.3}$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = \frac{1}{d(A_3, A^+)} = \frac{1}{0.19}$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = \frac{1}{d(A_4, A^+)} = \frac{1}{0.26}$$

and

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = \frac{1}{d(A_5, A^+)} = \frac{1}{0.29}$$

Then, it holds that $A_3 \succ A_4 \succ A_1 \sim A_5 \succ A_2$, and therefore, the alternative A_3 is still the preferred one.

Finally, if we look for the alternative that differs most from the worst possibility A^-, we can choose $\psi(x, y) = y$. In that case,

$$a_1 = \psi(d(A_1, A^+), d(A_1, A^-)) = d(A_1, A^-) = 0.34$$

$$a_2 = \psi(d(A_2, A^+), d(A_2, A^-)) = d(A_2, A^-) = 0.3$$

$$a_3 = \psi(d(A_3, A^+), d(A_3, A^-)) = d(A_3, A^-) = 0.38$$

$$a_4 = \psi(d(A_4, A^+), d(A_4, A^-)) = d(A_4, A^-) = 0.42$$

and

$$a_5 = \psi(d(A_5, A^+), d(A_5, A^-)) = d(A_5, A^-) = 0.28.$$

We conclude that $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$, whence A_4 is the best alternative. ■

The approach we have presented considers local AIF-divergences that allows us to compute the differences pointwise, and therefore, we can use different weights for different attributes/points. Furthermore, instead of using a function

$\psi(x, y) = \frac{x}{x+y}$, we define our procedure for a generic function ψ that can be defined by the expert depending on his/her preferences.

VII. Concluding Remarks

In our previous paper [26], we have introduced and investigated measures of comparison of AIF-sets named AIF-divergences. We have also justified why we considered AIF-divergences more adequate measures of comparison of AIF-sets than dissimilarities or distances.

In this paper, we have focused on a family of AIF-divergences that satisfies a local property. This local property allows us to define and compute the difference between two AIF-sets in a pointwise manner. This is an interesting property, and it is usually helpful for some application areas such as image processing, where "pointwise" computation can be understood as "pixel by pixel" computation.

After studying the properties of local AIF-divergences, we have demonstrated how this family of AIF-divergences can be applied in two different fields: pattern recognition and decision making. For pattern recognition, we can have applications where each attribute has an associated weight or importance. We have demonstrated how local divergence can effectively use this set of weights in decision making. This problem cannot be tackled with the usual approaches because they usually do not require the local property to define measures of comparison. Here, we have used local AIF-divergences to improve a method proposed in [32]. The original approach considered only four particular measures of comparisons, and none of them were AIF-divergences. Since we view AIF-divergences as more adequate measures of comparisons, we have adapted Xu's approach to use local AIF-divergences.

Two important problems arise from our results. First, some real-life problems are defined over weighted universes. For example, in pattern recognition, different features may have different relevance (weights) in decision making. Therefore, it would be interesting to investigate if it is possible to consider that weights in the definition of locality, in the sense that the difference between $D_{\mathrm{AIF}}(A, B)$ and $D_{\mathrm{AIF}}(A \cup \{x\}, B \cup \{x\})$ depends on the membership and nonmembership degrees of x to A and B as well as on the weight associated with x. Second, some authors have defined entropy measures for AIF-sets [7], [27], [29]. We think that local AIF-divergences could generate entropy measures just by imposing some additional properties. We leave these for our future investigation. In addition, we would like to develop applications of the local divergence measures in real-life large-scale pattern recognition problems.

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